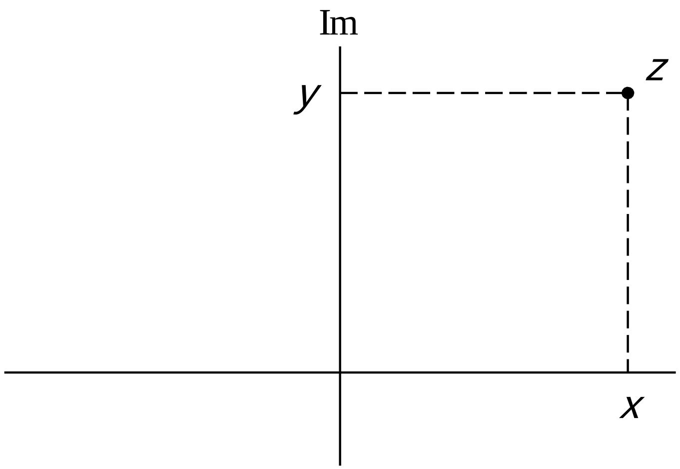
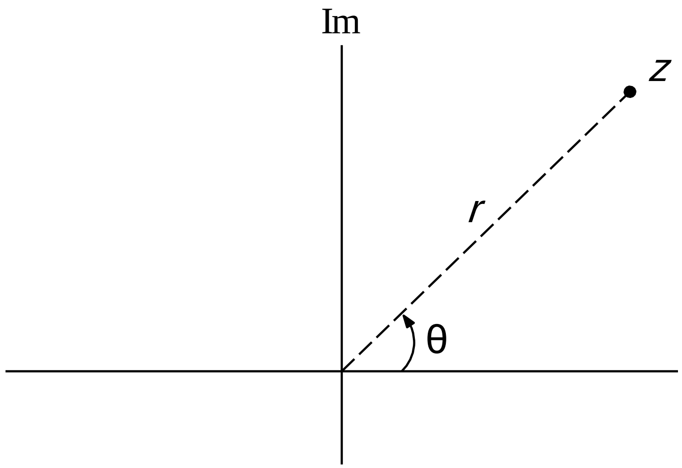
- Since $e^{j\theta} = e^{j(\theta + 2\pi k)}$ for all real θ and all integer k, the argument of a complex number is only uniquely determined to within an additive multiple of 2π.
- The principal argument of a complex number z, denoted $\operatorname{Arg} z$, is the particular value θ of $\arg z$ that satisfies $-\pi < \theta \leq \pi$.
- The principal argument of a complex number (excluding zero) is *unique*.





Cartesian form:

$$z = x + jy$$

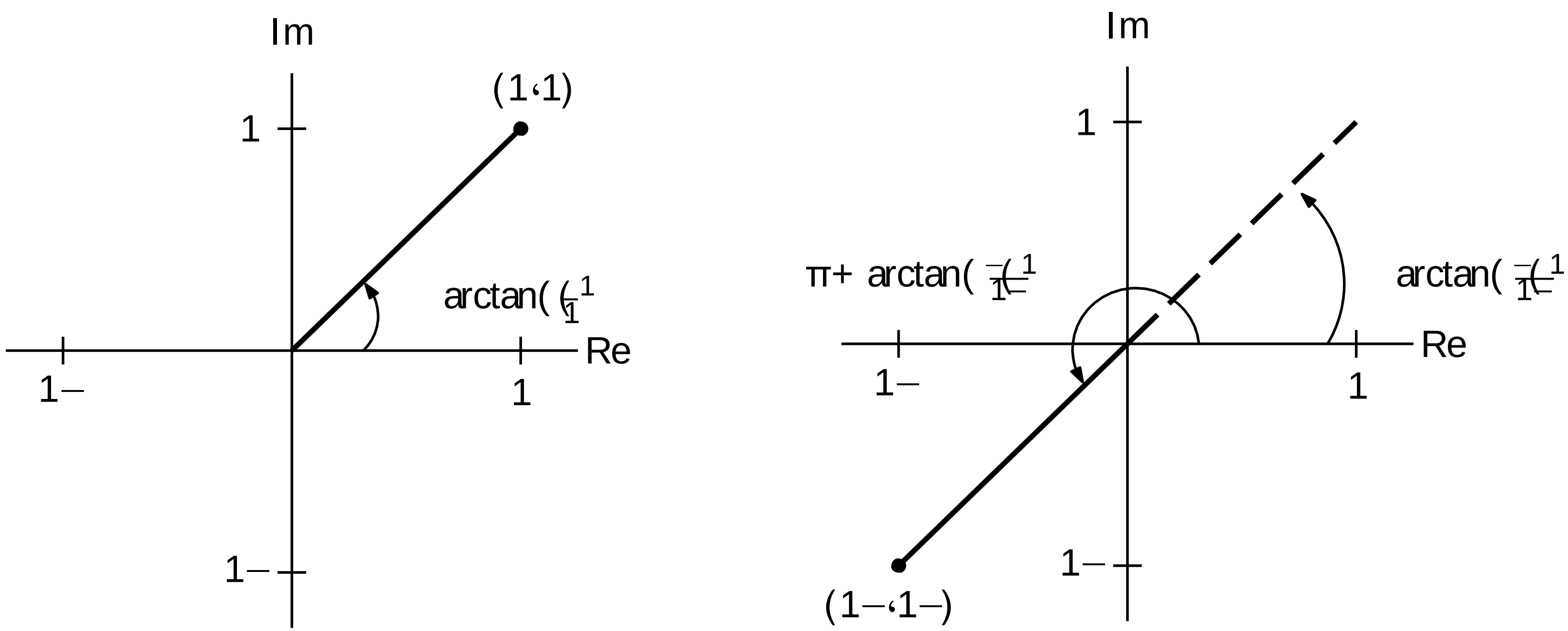
where $x = \mathrm{Re}\, z$ and $y = \mathrm{Im}\, z$

Polar form:

$$z = r(\cos\theta + j\sin\theta) = re^{j\theta}$$

where $r = |z|$ and $\theta = \arg z$

- The range of the arctan function is $-\pi/2$ (exclusive) to $\pi/2$ (exclusive).
- Consequently, the arctan function always yields an angle in either the first or fourth quadrant.



- The angle θ that a vector from the origin to the point (x, y) makes with the positive x axis is given by $\theta = \operatorname{atan2}(y, x)$, where

$$\operatorname{atan2}(y, x) = \begin{cases} \arctan(y/x) & \text{for } x > 0 \\ \pi/2 & \text{for } x = 0 \text{ and } y > 0 \\ -\pi/2 & \text{for } x = 0 \text{ and } y < 0 \\ \arctan(y/x) + \pi & \text{for } x < 0 \text{ and } y \geq 0 \\ \arctan(y/x) - \pi & \text{for } x < 0 \text{ and } y < 0. \end{cases}$$

- The range of the atan2 function is from $-\pi$ (exclusive) to π (inclusive).
- For the complex number z expressed in Cartesian form $x + jy$,
$\operatorname{Arg} z = \operatorname{atan2}(y, x)$.
- Although the atan2 function is quite useful for computing the principal argument (or argument) of a complex number, it is not advisable to memorize the definition of this function. It is better to simply understand what this function is doing (namely, intelligently applying the arctan function).

- Let z be a complex number with the Cartesian and polar form representations given respectively by

$$z = x + jy \quad \text{and} \quad z = re^{j\theta}.$$

- To convert from *polar to Cartesian* form, we use the following identities:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

- To convert from *Cartesian to polar* form, we use the following identities:

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \operatorname{atan2}(y, x) + 2\pi k,$$

  where k is an arbitrary integer.

- Since the atan2 function simply amounts to the intelligent application of the arctan function, instead of memorizing the definition of the atan2 function, one should simply *understand* how to use the arctan function to achieve the same result.

- For complex numbers, addition and multiplication are *commutative*. That is, for any two complex numbers z_1 and z_2,

$$z_1 + z_2 = z_2 + z_1 \quad \text{and}$$
$$z_1 z_2 = z_2 z_1.$$

- For complex numbers, addition and multiplication are *associative*. That is, for any two complex numbers z_1 and z_2,

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \quad \text{and}$$
$$(z_1 z_2) z_3 = z_1 (z_2 z_3).$$

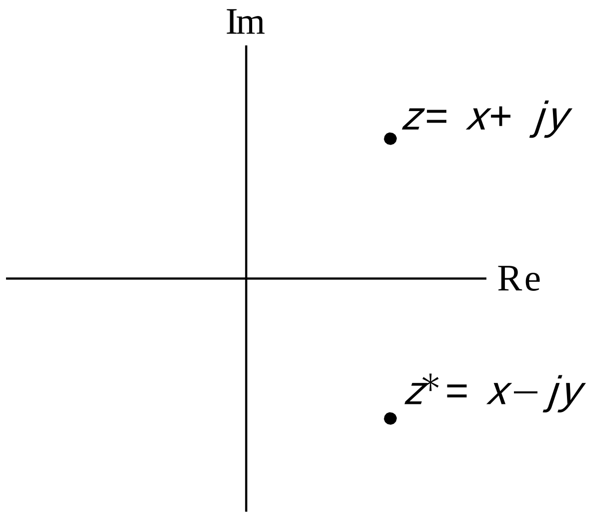
- For complex numbers, the *distributive* property holds. That is, for any three complex numbers z_1, z_2, and z_3,

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

- The conjugate of the complex number $z = x + jy$ is denoted as z^* and defined as

$$z^* = x - jy.$$

- Geometrically, the conjugation operation reflects a point in the complex plane about the real axis.
- The geometric interpretation of the conjugate is illustrated below.



- For every complex number z, the following identities hold:

$$
\begin{aligned}
|z^*| &= |z|, \\
\arg z^* &= -\arg z, \\
zz^* &= |z|^2, \\
\operatorname{Re} z &= \tfrac{1}{2}(z + z^*), \quad \text{and} \\
\operatorname{Im} z &= \tfrac{1}{2j}(z - z^*).
\end{aligned}
$$

- For all complex numbers z_1 and z_2, the following identities hold:

$$
\begin{aligned}
(z_1 + z_2)^* &= z_1^* + z_2^*, \\
(z_1 z_2)^* &= z_1^* z_2^*, \quad \text{and} \\
(z_1 / z_2)^* &= z_1^* / z_2^*
\end{aligned}
$$

- *Cartesian form:* Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$. Then,

$$\begin{aligned} z_1 + z_2 &= (x_1 + jy_1) + (x_2 + jy_2) \\ &= (x_1 + x_2) + j(y_1 + y_2). \end{aligned}$$

- That is, to add complex numbers expressed in Cartesian form, we simply add their real parts and add their imaginary parts.
- *Polar form:* Let $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$. Then,

$$\begin{aligned} z_1 + z_2 &= r_1 e^{j\theta_1} + r_2 e^{j\theta_2} \\ &= (r_1 \cos\theta_1 + jr_1 \sin\theta_1) + (r_2 \cos\theta_2 + jr_2 \sin\theta_2) \\ &= (r_1 \cos\theta_1 + r_2 \cos\theta_2) + j(r_1 \sin\theta_1 + r_2 \sin\theta_2). \end{aligned}$$

- That is, to add complex numbers expressed in polar form, we first rewrite them in Cartesian form, and then add their real parts and add their imaginary parts.
- For the purposes of addition, it is easier to work with complex numbers expressed in Cartesian form.

- *Cartesian form:* Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$. Then,

$$\begin{aligned} z_1 z_2 &= (x_1 + jy_1)(x_2 + jy_2) \\ &= x_1 x_2 + jx_1 y_2 + jx_2 y_1 - y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1). \end{aligned}$$

- That is, to multiply two complex numbers expressed in Cartesian form, we use the distributive law along with the fact that $j^2 = -1$.
- *Polar form:* Let $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$. Then,

$$z_1 z_2 = \left(r_1 e^{j\theta_1}\right)\left(r_2 e^{j\theta_2}\right) = r_1 r_2 e^{j(\theta_1 + \theta_2)}.$$

- That is, to multiply two complex numbers expressed in polar form, we use exponent rules.
- For the purposes of multiplication, it is easier to work with complex numbers expressed in polar form.

- *Cartesian form:* Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$. Then,

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{z_1 z_2^*}{|z_2|^2} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{x_2^2 + y_2^2}$$

$$= \frac{x_1x_2 - jx_1y_2 + jx_2y_1 + y_1y_2}{x_2^2 + y_2^2} = \frac{x_1x_2 + y_1y_2 + j(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}.$$

- That is, to compute the quotient of two complex numbers expressed in Cartesian form, we convert the problem into one of division by a real number.
- *Polar form:* Let $z_1 = r_1e^{j\theta_1}$ and $z_2 = r_2e^{j\theta_2}$. Then,

$$\frac{z_1}{z_2} = \frac{r_1e^{j\theta_1}}{r_2e^{j\theta_2}} = \frac{r_1}{r_2}e^{j(\theta_1-\theta_2)}.$$

- That is, to compute the quotient of two complex numbers expressed in polar form, we use exponent rules.
- For the purposes of division, it is easier to work with complex numbers expressed in polar form.

- For any complex numbers z_1 and z_2, the following identities hold:

$$|z_1 z_2| = |z_1||z_2|,$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \quad \text{for } z_2 \neq 0,$$

$$\arg z_1 z_2 = \arg z_1 + \arg z_2, \quad \text{and}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 \quad \text{for } z_2 \neq 0.$$

- The above properties trivially follow from the polar representation of complex numbers.

- Euler's relation. For all real θ,

$$e^{j\theta} = \cos\theta + j\sin\theta.$$

- From Euler's relation, we can deduce the following useful identities:

$$\cos\theta = \tfrac{1}{2}(e^{j\theta} + e^{-j\theta}) \quad \text{and}$$
$$\sin\theta = \tfrac{1}{2j}(e^{j\theta} - e^{-j\theta}).$$

- De Moivre's theorem. For all real θ and all *integer* n,

$$e^{jn\theta} = \left(e^{j\theta}\right)^n.$$

[Note: This relationship does not necessarily hold for *real* n.]

- Every complex number $z = re^{i\theta}$ (where $r = |z|$ and $\theta = \arg z$) has n distinct *nth roots* given by

$$\sqrt[n]{r}e^{i(\theta + 2\pi k)/n} \quad \text{for } k = 0, 1, \ldots, n-1.$$

- For example, 1 has the two distinct square roots 1 and -1.

- Consider the equation

$$az^2 + bz + c = 0,$$

where a, b, and c are real, z is complex, and $a \neq 0$.

- The roots of this equation are given by

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- This formula is often useful in factoring quadratic polynomials.
- The quadratic $az^2 + bz + c$ can be factored as $a(z - z_0)(z - z_1)$, where

$$z_0 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad z_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$